Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Group of Turns and Slides and the Geometry of Turbines.[*]

BY EDWARD KASNER.

We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A *turn* T_α converts each element into one having the same point and a direction making a fixed angle α with the original direction. By a *slide* S_k the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together generate a continuous group of three parameters which we denote by G_3.[†]

Applied to a simply infinite system of curves, the operation T_α produces a system of isogonal trajectories, while S_k produces equitangential trajectories. Repeating the operations, *i. e.* applying G_3, we obtain from a given system in general ∞^3 new systems. Certain systems, namely those with automorphic transformations, produce fewer trajectorial systems. These are readily determined.

If any transformation G_3 is applied to the ∞^1 elements of a point, the new ∞^1 elements form a configuration which we term a *turbine.* A turbine consists of ∞^1 elements whose points form a circle and whose directions are equally inclined to that circle. The transformations G_3 obviously convert turbines into turbines. There exists a larger group G_{15} of fifteen parameters having this property.

The geometry of turbines is most readily handled by means of a certain representation between the oriented elements of the plane and the points of three-dimensional space, in which turbines correspond to the straight lines of space. Those turbines which are unions, namely, the ∞^3 (oriented) circles, correspond to the lines of a certain linear complex L. In Lie's famous representation of circles by the lines of a complex, the lines not belonging to the complex are not considered; so the concept of turbine does not appear. Furthermore our representation is one-to-one, while Lie's, which employs non-oriented elements, is one-to-two.

---

[*] Read before the American Mathematical Society, December, 1908.

[†] In any space of constant curvature an analogous group, isomorphic with the displacement group of the space, may be obtained. Thus, for ordinary space the requisite element is the *feuillet*, consisting of a point, line, and plane all incident with one another, and there are three fundamental operations. In a space of variable curvature the operations on the elements generate a group involving in all probability an infinite number of parameters.

We next consider the analogue in the plane of configurations in space which are polar or conjugate with respect to L or the related null-system. In particular we are led to the concept of conjugate differential equations or systems of curves in the plane. This is made use of to prove quite simply and naturally Scheffers' theorems on isogonal and equitangential trajectories.* These results are included as special cases of general theorems (numbered 8 and 9 below) concerning systems derived by applying any one-parameter subgroup of G_3, instead of the particular subgroups S and T.

§1. *The Group G_3.*

All turns T_α constitute a one-parameter group T; and all slides S_k a one-parameter group S. We now prove

THEOREM 1. *All turns and slides generate a three-parameter group G_3.*

We note in the first place that if any two successions of turns and slides have the same effect upon one lineal element they will have the same effect upon all elements. In the second place we may convert a given element into any other element by a succession of the type

$$T_\alpha S_k T_\beta. \qquad (1)$$

Hence a succession of any number of turns and slides will be equivalent to some succession (1). The parameters α, k, β are clearly essential; hence the theorem stated.

For the analytic representation, it will be convenient to define an element, not by the usual (x, y, y'), but by the coordinates (u, v, v') employed by Scheffers: v is the perpendicular from the origin, u is the angle between the perpendicular and the initial line, and the derivative $v' = dv/du$, which we shall also denote by w, is the distance between the foot of the perpendicular and the point of the element.

---

* *Mathematische Annalen*, Vol. LX (1905).

The slide S_k is then $u_1 = u$, $v_1 = v$, $w_1 = w + k$; we write it simply

$$u, \quad v, \quad w + k. \tag{2}$$

The turn T_α is

$$u + \alpha, \quad v \cos \alpha + w \sin \alpha, \quad - v \sin \alpha + w \cos \alpha. \tag{3}$$

The product (1) is then

$$\begin{aligned} u + \alpha + \beta, \quad v \cos (\alpha + \beta) + w \sin (\alpha + \beta) + k \sin \beta, \\ - v \sin (\alpha + \beta) + w \cos (\alpha + \beta) + k \cos \beta. \end{aligned} \tag{4}$$

Introducing new parameters, we may write our group G_3 in the form

$$u + \gamma, \quad v \cos \gamma + w \sin \gamma + a, \quad - v \sin \gamma + w \cos \gamma + b. \tag{5}$$

Since turns and slides are not contact transformations, it is clear that the transformations (5) will not usually be contact transformations. The conditions for a contact transformation give $b = 0$, and $\sin \gamma = 0$; that is, $k \cos \beta = 0$ and $\alpha + \beta = 0$ or π. The resulting transformations are of one of the forms

$$T_{\frac{\pi}{2}} S_k T_{-\frac{\pi}{2}}, \quad T_{\frac{\pi}{2}} S_k T_{\frac{\pi}{2}}.$$

The first represents a dilatation D_k; and the second, which may be written $D_k T_\pi$, represents a dilatation accompanied by reversal of orientation.

Hence the only contact transformations in the group G_3 *are dilatations, and dilatations with reversal of orientation.*

Our group may be written in the simple form

$$S_k D_d T_\alpha. \tag{6}$$

As the independent infinitesimal transformations generating the group, we may take the infinitesimal slide, dilatation, and turn. The symbols of these generators in the Lie notation are respectively

$$\frac{\partial}{\partial w}, \quad \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w}. \tag{7}$$

In terms of the parameters k, d, α in (6), the multiplication of two transformations of the group G_3 is given by the same formulas as arise in the combination of two plane displacements with the usual parameters (x_0, y_0, θ). It is of course easy to see that the two groups are simply isomorphic: each may be regarded as a "group of parameters" in relation to the other group.

Corresponding to the translation subgroup, we have in G_3

$$G_2 \equiv S_k D_d, \tag{8}$$

which is therefore an invariant subgroup. By these transformations elements are converted into parallel elements, the sense being also preserved. It may be noted that any operation of G_3 is commutative with any displacement.

§ 2. *Systems of Curves with Auto-transformations.*

As a simple application, we may readily find all systems of ∞^1 curves which admit an infinitesimal transformation of the group G_3.

From (7) the general infinitesimal transformation is

$$a\frac{\partial}{\partial u} + (aw + b)\frac{\partial}{\partial v} + (c - av)\frac{\partial}{\partial w}. \tag{9}$$

If this is to leave invariant the system whose differential equation, in line coordinates, is $w = \frac{dv}{du} = f(u, v)$, then

$$af_u + (af + b)f_v + av - c = 0.$$

The general integration may be avoided by reducing to certain canonical forms.

First assume $a \neq 0$. Then by means of transformations S and D we may reduce (9) to $\frac{\partial}{\partial u} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}$; that is, to a turn. The invariant system must then consist of ∞^1 points. Applying D and S reversed we see that the required system consists of equitangentials of ∞^1 congruent circles; or, what is the same, of isogonals of a set of congruent circles. Next assume $a = 0$. If also $b = 0$, the transformation (9) is simply a slide and the required system consists of ∞^1 straight lines. If $b \neq 0$, it may be reduced to $b = 0$ by means of a turn. Hence the required system consists of isogonals of a set of straight lines.

THEOREM 2. *The only systems of ∞^1 curves which admit an infinitesimal transformation of the group G_3 are those composed of the isogonals, for a fixed angle, of any set of equal circles (including the case of any set of straight lines).*

Given a system of ∞^1 curves, and applying the constructions for isogonal and equitangential trajectories in all possible combinations, we obtain in general ∞^3 new systems. The systems described in the above theorem are distinguished by the fact that they possess fewer than ∞^3 derived systems.*

---

*The usual number of derived systems is ∞^2. The number will reduce to ∞^1 if the original system allows two infinitesimal transformations. The only real system of this kind is that composed of parallel straight lines.

Our results may be obtained synthetically by considering the action of the infinitesimal G_3 transformation or the one-parameter group generated by it. Each element is converted successively into the elements of a turbine. In the case of T the paths are points, and in the case of S they are straight lines. *In the general case it is easily seen that the paths are* ∞^2 *congruent turbines.* The required differential equations are generated by ∞^1 of these turbines.

§ 3. *The General Turbine Group* G_{15}.

A turbine is defined as a series of ∞^1 elements obtained from the elements of an oriented circle by means of a turn. The equations of a turbine are of the form

$$v = A \cos u + B \sin u + C, \quad w = B \cos u - A \sin u + D. \tag{10}$$

The result is a circle if $D = 0$. It is a point if $C = D = 0$. If the base circle of the turbine is a straight line, the analytic representation becomes

$$u = c_0, \quad c_1 v + c_2 w + c_3 = 0.$$

Both turns and slides convert turbines into turbines. We now prove

THEOREM 3. *All element transformations converting turbines into turbines constitute a fifteen-parameter group* G_{15}.

The direct attack is rather long but not difficult. The differential equations defining the ∞^4 turbines are

$$\frac{d^2 v}{du^2} = \frac{dw}{du}, \quad \frac{d^2 w}{du^2} = -\frac{dv}{du}. \tag{11}$$

Expressing the invariance of this set under an infinitesimal transformation

$$\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \zeta \frac{\partial}{\partial w}, \tag{12}$$

we find a set of partial differential equations for ξ, η, ζ whose solution gives

$$\begin{aligned}
\xi &= [2\alpha v + 2\beta w + a_2 - b_1] \cos u + [2\beta v - 2\alpha w \\
&\qquad + a_1 + b_2] \sin u + 2\gamma v + 2\delta w + d_1, \\
\eta &= [\beta(v^2 + w^2) + a_1 v + a_2 w + a_3] \cos u + [-\alpha(v^2 + w^2) \\
&\qquad + b_1 v + b_2 w + b_3] \sin u + \delta(w^2 - v^2) + 2\gamma v w + c_1 v + c_2 w + d_2, \\
\zeta &= [-\alpha(v^2 + w^2) + b_1 v + b_2 w + b_3] \cos u - [\beta(v^2 + w^2) \\
&\qquad + a_1 v + a_2 w + a_3] \sin u + \gamma(w^2 - v^2) - 2\delta v w - c_2 v + c_1 w + d_3;
\end{aligned} \tag{13}$$

involving fifteen parameters.

The group G_3 is of course a subgroup of G_{15}. The contact transformations in G_{15} constitute the G_{10} converting circles into circles.

§ 4. *Representation in Space.*

The group G_{15} may be determined most readily by means of a certain new set of coordinates for oriented elements, namely

$$X = e^{iu}, \quad Y = v - iw, \quad Z = (v + iw)e^{iu}. \tag{14}$$

The equations of the turbine (10) then become linear,

$$Y = \lambda_0 X + \mu, \quad Z = \mu_0 X + \lambda, \tag{15}$$

where

$$\lambda = A + iB, \quad \lambda_0 = A - iB, \quad \mu = C + iD, \quad \mu_0 = C - iD;$$

so that for real turbines λ, λ_0 and μ, μ_0 are pairs of conjugate complex numbers.

We may regard (14) as establishing a correspondence between the elements (u, v, w) of the plane and the points (X, Y, Z) of ordinary space in cartesian coordinates. *In this representation, which we designate by R, the ∞^4 turbines of the plane are pictured by the ∞^4 straight lines of space* (THEOREM 4).

It follows that element transformations converting turbines into turbines must correspond to collineations in space and hence constitute a continuous group G_{15}.

If we define points in space by homogeneous coordinates (X_1, X_2, X_3, X_4), we may write the fundamental representation R in the more symmetric form

$$X_1 : X_2 : X_3 : X_4 = e^{\frac{1}{2}iu} : (v - iw)e^{-\frac{1}{2}iu} : (v + iw)e^{\frac{1}{2}iu} : e^{-\frac{1}{2}iu}. \tag{14'}$$

The element corresponding to a given point of space is

$$u = -i \log X, \quad v = \frac{1}{2}\left(Y + \frac{Z}{X}\right), \quad w = \frac{i}{2}\left(Y - \frac{Z}{X}\right).$$

Therefore if we consider only the finite elements of the plane and exclude the points for which $X = 0$, the correspondence R is one-to-one.*

To an arbitrary curve in space corresponds an arbitrary series of ∞^1 elements in the plane. The series will constitute a union if $dv - w\,du = 0$; that is, if

$$dZ + X dY - Y dX = 0. \tag{16}$$

---

* The representation R is closely related and essentially equivalent to the familiar representation of Lie's in which the circles of the plane correspond to the lines of a linear complex. Cf. Lie-Scheffers, "Berührungstransformationen," p. 249. Lie, however, does not orient the elements, so that his correspondence is one-to-two; and turbines are not introduced, so that only lines belonging to the linear complex are represented in the plane.

This Monge equation defines a certain null-system in space. We obtain the same null-system by examining the ∞^3 straight lines of space corresponding to the ∞^3 circles of a plane. The condition that the turbine (14) shall reduce to a circle is $D = 0$; hence the corresponding straight line (15) satisfies the condition $\mu = \mu_0$. This defines a certain linear complex of lines, which we denote by L. The related null-system is of course (16).

THEOREM 5. *Unions in the plane are represented in space by the curves satisfying* (16); *that is, curves whose tangents belong to the fundamental linear complex L. The lines of this complex correspond to the circles of the plane.*

§ 5. *The Operation N. Conjugate Configurations.*

The fundamental linear complex L defines a correspondence such that to each point corresponds a certain plane passing through it (null-system). If any configuration is given in space we may construct the polar or conjugate configuration. Employing our representation R we obtain two configurations in the plane which we also term conjugate. The passage from a plane configuration to its conjugate, we describe as operation N.

The polar of a point in space is a plane passing through it. Hence the conjugate of a given lineal element consists of ∞^2 elements cocircular with it.

Conjugate straight lines in space lead to conjugate turbines. These have the same circle as point locus and the elements are symmetrically related to the elements of the circle. The self-conjugate turbines are of course simply circles.

The conjugate of any series of elements is a new series of elements each of which is cocircular with three consecutive elements of the given series; and vice versa, since the operation N is involutorial.

To a surface in space corresponds by R a field of ∞^2 elements in the plane; that is, a differential equation of the first order with its system of ∞^1 integral curves. If the surface is a plane its image is a parabolic pencil of circles. All the points of an arbitrary surface in the neighborhood of one of its points lie in a plane (tangent plane). *Hence all the elements of an arbitrary field or system of* ∞^1 *plane curves are cocircular with some definite element,* that is, belong to a definite parabolic pencil of circles; this may be called the approximating or tangent pencil.

We thus arrive at the important concept of conjugate differential equations of the first order, or conjugate systems of curves.

THEOREM 6. *To a general system Σ of ∞^1 curves in the plane corresponds by the operation N a definite conjugate system $\bar{\Sigma}$. Those elements of Σ in the infinitesimal neighborhood of any one of the elements of Σ are cocircular with one of the elements of $\bar{\Sigma}$; and vice versa, the relation being mutual.*[*]

The only exception arises in the case where the system Σ has for its image in space a developable surface. The field of elements in this case is generated by ∞^1 turbines each having an element in common with the consecutive turbine. Such a field is peculiar in that the conjugate configuration is not a field but merely a series of ∞^1 elements. A still more special case arises when the image surface is a plane. The field then consists of all the elements cocircular with a fixed element; that is, Σ is a parabolic pencil of circles. In this case the conjugate configuration reduces to the fixed element.

The conjugate of a differential equation, $f(u, v, v') = 0$, is most readily derived by introducing the coordinates X, Y, Z. If the given equation then takes the form $Z = F(X, Y)$, the conjugate equation is obtained in the form $\bar{Z} = \bar{F}(\bar{X}, \bar{Y})$ by eliminating X and Y from

$$\bar{X} = -F_Y, \quad \bar{Y} = F_X, \quad \bar{Z} = F - XF_X - YF_Y, \tag{17}$$

relations which follow immediately from (19) below.

For any collineation of space there is a definite conjugate collineation with respect to the null-system. Hence for any element transformation G of the group G_{15} of § 3 there is a conjugate transformation

$$\bar{G} = N^{-1}GN = NGN \tag{18}$$

also in the group. If C and $\bar{C}$ are conjugate configurations, so also are CG and $\bar{C}\bar{G}$. The determination by a construction in the plane follows from the fact that if e and e' are cocircular elements, so also are eG and $e'\bar{G}$. We now prove

THEOREM 7. *The conjugate of a transformation belonging to the group G_8 generated by turns and slides, also belongs to that group.*

---

[*] If we consider a congruence of lines belonging to L, we see that the two focal surfaces are conjugate with respect to L. Hence in the plane a general congruence of ∞^2 circles may be regarded as composed of the circles of curvature of two distinct systems of ∞^1 curves, Σ and $\bar{\Sigma}$. Cf. Liebmann's discussion in the new edition of Pascal's *Repertorium*, Vol. II, p. 500.

For this purpose we write our polarity in the form of a transformation from the surface element (X, Y, Z, P, Q) to the surface element

$$-Q,\ P,\ Z-XP-YQ,\ Y,\ -X. \tag{19}$$

In the same coordinates the transformation G_3 is

$$cX,\ cY+c',\ Z+c''X,\ \frac{1}{c}P+\frac{c''}{c},\ \frac{1}{c}Q. \tag{20}$$

Calculating $\overline{G}$ from (18), we find that it has the form (20) with new parameters

$$\bar{c}=1/c,\quad \bar{c}'=c''/c,\quad \bar{c}''=c'/c. \tag{21}$$

In terms of the form (6) for the group G_3, it may be shown directly that the *conjugate of the transformation* $S_k D_d T_\alpha$ *is* $S_{-k} D_d T_{-\alpha}$.

In particular the conjugate of T_α is $T_{-\alpha}$, and of S_k is S_{-k}. Hence if two systems of curves are isogonally or equitangentially related, the same will be true of the two conjugate sets, results equivalent to those of Scheffers. A similar result holds for two systems related by dilatation.

§ 6. *Generalization of Scheffers' Theory of Trajectories.*

Consider a general system Σ of ∞^1 oriented curves. Scheffers applies the one-parameter group of turns on the one hand, obtaining ∞^1 systems (isogonals), and the one-parameter group of slides on the other hand, obtaining ∞^1 new systems (equitangentials). The two sets of results relating to circles of curvature and reciprocity relations for the doubly-infinite systems thus derived are written in parallel columns and a certain (non-projective) duality appears. We may, from our point of view, obtain these results simultaneously as special cases of one theory by starting with *any* one-parameter subgroup G_1 of the group G_3. Applying G_1 to the given system Σ, we obtain ∞^1 new systems or, collectively, a doubly-infinite system Σ_2.*

For the group G_1 the path of each element of the plane is a turbine K. There are ∞^2 of these turbines, all congruent: we denote the totality by K_2. To state the results it is necessary to consider also the conjugate group $\overline{G}_1$. The path turbines for this will form a second set of ∞^2 congruent turbines $\overline{K}_2$. The two sets of turbines are built on equal circles, but the angles at which the elements are inclined to the circles, while equal in magnitude, are on opposite sides of the elements of the circles.

---

* It is assumed of course that the given system Σ is not invariant under G_1.

The general results for any group G_1, which are obtained synthetically or analytically without difficulty, are as follows:

THEOREM 8. *Consider any one of the path turbines K of the set K_2 connected with G_1. Each element of K determines a curve of the doubly-infinite system Σ_2 generated by applying G_1 to any simply-infinite system Σ. The locus of the centers of the ∞^1 circles osculating these curves at these elements is a straight line.*[*] *These ∞^1 circles hence touch a certain turbine K' of the set $\overline{K}_2$ conjugate to K_2.*[†]

THEOREM 9. *According to the previous theorem, the system Σ_2 obtained by applying G_1 to Σ induces a definite correspondence between the set of turbines K_2 and the conjugate set $\overline{K}_2$. There exists another system $\overline{\Sigma}_2$, obtained by applying $\overline{G}_1$ to $\overline{\Sigma}$, for which this correspondence is precisely reversed.*

For the group of turns (or slides), the sets K_2 and $\overline{K}_2$ are points (or lines), and we obtain Scheffers' results for isogonals (or equitangentials). It is to be observed that, in the general case, the sets of ∞^1 osculating circles are linear in the sense of Lie's (higher) geometry of circles, but not usually in the sense of elementary circle geometry.

COLUMBIA UNIVERSITY, NEW YORK.

---

[*] A single exception arises when the group G_1 is the dilatation group D. The ∞^1 osculating circles are then obviously concentric.

[†] To a given K will correspond, not its own conjugate $\overline{K}$, but *some* member of the conjugate set $\overline{K}_2$. Just which member, is determined by the curves Σ_2.